

On Mostow rigidity for variable negative curvature

Igor Belegradek

Abstract

We prove a finiteness theorem for the class of complete finite volume Riemannian manifolds with pinched negative sectional curvature, fixed fundamental group, and of dimension ≥ 3 . One of the key ingredients is that the fundamental group of such a manifold does not admit a small nontrivial action on an \mathbf{R} -tree.

1 Introduction

According to the Mostow rigidity theorem, the isometry type of a complete finite volume locally symmetric negatively curved Riemannian n -manifold with $n \geq 3$ is uniquely determined by its fundamental group. This is no longer true for finite volume manifolds of variable sectional curvature. In fact, there even exist negatively curved manifolds which are homeomorphic but not diffeomorphic to finite volume hyperbolic manifolds [FJO98]. Yet it turns out that there are essentially finitely many possibilities for the geometry and topology of such manifolds provided the sectional curvature is pinched between two negative fixed constants.

For reals $a \leq b \leq 0$, let $\mathcal{M}_{a,b,\pi,n}$ be the class of complete finite volume Riemannian manifolds of dimension $n \geq 3$ with sectional curvatures in $[a, b]$ and fundamental groups isomorphic to π . Note that n can be read off the fundamental group π , namely, $n = \max(\text{cd}(\pi), \text{cd}(N) + 1)$ where “cd” is the cohomological dimension and N is a maximal nilpotent subgroup of π . Also, $\text{cd}(\pi) = n$ iff each manifold in $\mathcal{M}_{a,b,\pi,n}$ is closed. Here is our main result.

Theorem 1.1. *The class $\mathcal{M}_{a,b,\pi,n}$ falls into finitely many diffeomorphism types. Furthermore, for any sequence of manifolds in $\mathcal{M}_{a,b,\pi,n}$, there exists a subsequence (M_k, g_k) , a smooth manifold M , and diffeomorphisms $f_k : M \rightarrow M_k$ such that the pullback metrics $f_k^* g_k$ converge in $C^{1,\alpha}$ topology uniformly on compact subsets to a complete finite volume $C^{1,\alpha}$ Riemannian metric on M .*

Although this result does not appear in the literature, it is not new except in dimension 4 where only homeomorphism finiteness has been known. However, the proof we present is very different from the existing argument that runs as follows.

M. Gromov and W. Thurston [Gro82] used straightening and bounded cohomology to deduce that volume is bounded by the simplicial volume:

$$\text{vol}(M) \leq C(a/b, n) \|M\| < \infty$$

for any $M \in \mathcal{M}_{a,b,\pi,n}$. Since all the manifolds in $\mathcal{M}_{a,b,\pi,n}$ belong to the same proper homotopy type, they must have equal simplicial volumes. Thus, volume is uniformly bounded from above on $\mathcal{M}_{a,b,\pi,n}$. Furthermore, a universal lower volume bound comes from the Heintze-Margulis theorem [BGS85]. For closed manifolds of dimension $n \geq 4$ and sectional curvatures within $[-1, 0)$, the diameter can be bounded in terms of volume [Gro78]. Hence the conclusion of 1.1 follows from the Cheeger-Gromov compactness theorem. Similarly, the conclusion of the theorem 1.1 for finite volume manifolds of dimension ≥ 5 can be deduced from the work of K. Fukaya [Fuk84]. The dimension restriction comes from treating the ends by the weak h-cobordism theorem. In fact, Fukaya proves a similar statement for $n = 4$ where diffeomorphism finiteness is replaced by homotopy finiteness. (The topological 4-dimensional weak h -cobordism theorem, unavailable at the time of [Fuk84], can be also applied here because we deal with virtually nilpotent fundamental groups [Gui92, FT95]. Yet this only gives homeomorphism rather than diffeomorphism finiteness.)

By contrast, main ideas in our approach come from Kleinian groups and geometric group theory. Essentially, given a degenerating sequence of manifolds M_k , one can use rescaling in the universal covers to produce a nontrivial action of π on an \mathbf{R} -tree with virtually nilpotent arc stabilizers (cf. [Bes88, Pau88, Pau91]). Then results of Rips, Bestvina and Feighn [BF95] imply that π splits over a virtually nilpotent subgroup. We prove that this does not happen if $\mathcal{M}_{a,b,\pi,n}$ is nonempty. Then the methods of the Cheeger-Gromov compactness theorem imply that M_k subconverges in pointed $C^{1,\alpha}$ topology to a complete $C^{1,\alpha}$ Riemannian manifold M . We then prove that $\pi_1(M)$ contains a subgroup isomorphic to π which implies that M has finite volume and, in fact, $\pi_1(M) \cong \pi$. Now the convergence $M_k \rightarrow M$ is analogous to strong convergence of Kleinian groups. Studying this convergence yields the theorem 1.1. Instead of using the h-cobordism theorem to deal with ends, we find a direct geometric argument that works in all dimensions. Similarly to [Gro82], our methods provide a uniform upper bound on the volume of manifolds in $\mathcal{M}_{a,b,\pi,n}$.

Note that the real Schwarz lemma of Besson-Courtois-Gallot [BCG98] gives yet

another way to get a uniform upper bound on volume of *compact* manifolds in $\mathcal{M}_{a,b,\pi,n}$, and hence another proof of 1.1 in this case. It is still an open question whether their method extends to finite volume manifolds.

Technically, our proof is much easier for closed manifolds; this case was previously treated in [Bel99]. One of the key facts needed for the theorem 1.1 is the following lemma which is of independent interest.

Lemma 1.2. *If the class $\mathcal{M}_{a,b,\pi,n}$ is nonempty, then π does not split over a virtually nilpotent subgroup. Furthermore, $\text{Out}(\pi)$ is finite and π is cohopfian.*

That π does not split over a virtually nilpotent subgroup can be also deduced (after some extra work) from a recent paper of B. Bowditch [Bow98] who studied the structure of the splittings of relatively hyperbolic groups over subgroups of peripheral groups. Unlike Bowditch's work that applies in a more general situation, our argument is completely elementary.

Recall that a group π is called cohopfian if it has no proper subgroups isomorphic to π . That $\text{Out}(\pi)$ is finite and π is cohopfian is due to G. Prasad [Pra76] in locally symmetric case. Here we follow the idea of F. Paulin [Pau91], and E. Rips and Z. Sela [RS94] who proved these properties for a large class of word-hyperbolic groups.

Topology of complete finite volume negatively curved manifolds seems to be encoded in the fundamental group. It is a deep recent result of F. T. Farrell and L. Jones [FJ93b, FJ98] that any homotopy equivalence of two manifolds in the class $\mathcal{M}_{n,a,b,\pi}$ with $n \geq 5$ is homotopic to a homeomorphism (which is a diffeomorphism away from compact subsets). Then the smoothing theory [HM74, KS77] implies that there exist at most finitely many non-diffeomorphic manifolds in the class $\mathcal{M}_{a,b,\pi,n}$ with $n \geq 5$. Furthermore, if $n \geq 6$, there are homeomorphic negatively curved manifolds that are not diffeomorphic [FJ89b, FJ89a, FJ93a, FJO98]. Such examples are still unknown if $n = 4, 5$. If $n = 3$, one expects that any two manifolds in the class $\mathcal{M}_{a,b,\pi,n}$ are diffeomorphic. This is known for Haken manifolds [Wal68] (note that any noncompact finite volume manifold is Haken). For non-Haken manifolds this would follow from (as yet unproved) Thurston's hyperbolization conjecture and Mostow Rigidity.

Note that 1.1 gives diffeomorphism finiteness in all dimensions. As we explained above this is mostly interesting if $n = 4$; in fact, the interior of any compact smooth 4-manifold with nonempty boundary has at least countably many smooth structures [BE98].

Theorem 1.1 combined with results of Gao [Gao90] immediately implies that the class of finite volume Einstein manifolds that lie in $\mathcal{M}_{a,b,\pi,n}$ is compact in pointed C^∞ topology (i.e. for any sequence of Einstein manifolds in $\mathcal{M}_{a,b,\pi,n}$, there is a subsequence M_k , a manifold M , and diffeomorphisms $f_k: M \rightarrow M_k$ such that $f_k^* g_k$ converge in C^∞ topology uniformly on compact subsets to a complete finite volume Einstein metric on M). Furthermore, since negatively curved Einstein metrics on compact manifolds of dimension > 2 are isolated in the moduli space of the Einstein metrics [Bes88, 12.73], we conclude that, up to homothety, there are only finitely many compact Einstein manifolds in $\mathcal{M}_{a,b,\pi,n}$. It is a tantalizing open problem to decide whether any compact negatively curved Einstein manifold is locally symmetric. A related question was recently resolved in dimension four: any Einstein metric on a compact negatively curved locally symmetric 4-manifold is locally symmetric [BCG95, LeB95].

One can also use 1.1 to deduce several pinching results. For example, given a group π , there exists an $\epsilon = \epsilon(\pi) > 0$ such that any finite volume manifold from $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ is diffeomorphic to a real hyperbolic manifold. Note that ϵ has to depend on the topology of the manifolds (in our case on the fundamental group) as examples [FJ89b, FJ93a, FJO98, GT87] show. Similar results hold for almost quarter pinched Kähler and quaternionic-Kähler manifold manifolds.

The structure of the paper is as follows. In the section 2 we give some background on finite volume negatively curved manifolds. Sections 3 and 5 contain a proof of 1.2. Convergence of finite volume negatively curved manifolds is discussed in Section 4. Theorem 1.1 is proved in Section 6. Applications to pinching are discussed in Section 7.

The author is grateful to Christoph Böhm, Brian Bowditch, Harish Seshadri, Peter Shalen, Christopher Stark, and McKenzie Wang for helpful discussions or communications.

2 Preliminaries

Let M be a finite volume complete Riemannian manifold of sectional curvature within $[a, b]$, $a \leq b < 0$ of dimension $n > 2$. In this section we list some properties of M which we are going to use throughout the paper without explicit references. A comprehensive account on nonpositive curvature can be found in [BGS85]. Also see [HH77, Sch84, Bow93, Bow95].

2.1. Virtually nilpotent subgroups. Write M as X/π where X is the universal cover of M and $\pi \cong \pi_1(M)$ is the covering group. By the Cartan-Hadamard theorem X is diffeomorphic to the Euclidean space, thus M is

aspherical. An infinite order isometry γ of X either stabilizes a bi-infinite geodesic or fix exactly one point at infinity; such a γ is called *hyperbolic* or *parabolic*, respectively.

Virtually nilpotent discrete subgroups of $Isom(X)$ are finitely generated [Bow93]. Since X is contractible, π is torsion free. Any nontrivial virtually nilpotent subgroup of π is either an infinite cyclic group generated by a loxodromic isometry and stabilizing a bi-infinite geodesic, or a group that consists of parabolic isometries with a common fixed point at infinity. Any virtually nilpotent subgroup of π lies in a *unique* maximal virtually nilpotent subgroup of π . See [BGS85, Bow93, Bow95] for more information.

2.2. Thin/thick decomposition. For a positive ϵ , write $M_{[\epsilon, \infty)}$ for the ϵ -*thick part* of M which is the set of points of M with injectivity radius $\geq \epsilon$. Similarly, $M_{(0, \epsilon)} = M \setminus M_{[\epsilon, \infty)}$ is called the ϵ -*thin part* of M .

According to the Margulis lemma, there exists a universal constant $\mu_{n,a}$ such that for each $\epsilon < \mu_{n,a}$ the ϵ -thin part of M is a union of *finitely many* connected components. Unbounded components are called *cusps* while bounded components are called *tubes*. Each tube contains a closed geodesic of length $\leq 2\epsilon$ and is homeomorphic to the tubular neighborhood of the geodesic. Since there are only finitely many tubes, we can assume that $M_{(0, \epsilon)}$ consists of cusps by taking ϵ small enough.

Each cusp is a union of geodesic rays emanating from a common point at infinity. Also, given a cusp C , let \tilde{C} be a connected component of the preimage of C in the universal cover. The group $\{\gamma \in \pi : \gamma(\tilde{C}) = \tilde{C}\}$ coincides with the stabilizer Γ_z in Γ of a point z at infinity. The group Γ_z preserves horospheres centered at z and acts on each horosphere with compact quotient since $C = \tilde{C}/\Gamma_z$ has finite volume. Horospheres are C^2 submanifolds of X each diffeomorphic to the Euclidean space [HH77]. Each horosphere centered at z is orthogonal to geodesics asymptotic to z . Tangent vectors to such geodesics form the so-called *radial* vector field on X and C ; this is a C^1 -vector field [HH77]. (Throughout the paper all geodesic are assumed to have unit speed.) In fact, radial vector field is the gradient of the so called Busemann function. Any Busemann function defines a C^2 -Riemannian submersion of a cusp region bounded by a horosphere into the real line. Each cusp is diffeomorphic to the product of a real line and a closed aspherical manifold which is the quotient of a horosphere by Γ_z . Note that the boundary ∂C of a cusp is generally nonsmooth. Pushing along geodesic rays asymptotic to x defines a homeomorphism of ∂C and the Γ_z -quotient of a horosphere.

2.3. Compactification. According to [BGS85], M is diffeomorphic to the interior of a compact manifold with (possibly empty) boundary. If the boundary is nonempty, its connected components are quotients of horospheres by maximal parabolic subgroups of π . Each boundary component corresponds to a conjugacy class of maximal parabolic subgroups of π . The inclusion of each boundary component is π_1 -injective.

2.4. Exponential convergence of geodesics. To simplify notations we let $a = -K^2$, $b = -k^2$. Let γ_1, γ_2 be two geodesics asymptotic to a point $z = \gamma_1(+\infty) = \gamma_2(+\infty)$ such that $\gamma_1(0), \gamma_2(0)$ lie on the same horosphere centered at z . Then for any t , $\gamma_1(t), \gamma_2(t)$ lie on the same horosphere centered at z . Denote by $h(t)$ the distance between $\gamma_1(t)$ and $\gamma_2(t)$ on this horosphere equipped with the induced Riemannian metric. Also denote by $d(t)$ the distance between $\gamma_1(t)$ and $\gamma_2(t)$ in X . It is proved in [HH77] that for $t \geq 0$

$$e^{kt} \leq \frac{h(0)}{h(t)} \leq e^{Kt} \quad \text{and} \quad \frac{2}{k} \sinh\left(\frac{kd(t)}{2}\right) \leq h(t) \leq \frac{2}{K} \sinh\left(\frac{Kd(t)}{2}\right).$$

Therefore, we deduce that $d(t) \leq h(t) \leq h(0)e^{-kt} \leq \frac{2}{K} \sinh\left(\frac{Kd(0)}{2}\right) e^{-kt}$ and $d(0)e^{-Kt} \leq h(0)e^{-Kt} \leq h(t) \leq \frac{2}{K} \sinh\left(\frac{Kd(t)}{2}\right)$ for all $t \geq 0$. It is straightforward to check that $\sinh(x) \leq 2x$ whenever $x \in [0, 1]$. Also, it follows from comparison with Euclidean triangles that $d(t) \leq d(0)$ for $t \geq 0$. Hence, if $d(0) \leq 2/K$ and $t \geq 0$, then $e^{kt}/2 \leq d(0)/d(t) \leq 2e^{Kt}$.

3 No splitting

Throughout this section π is the fundamental group of a finite volume noncompact complete Riemannian manifold M of dimension $n > 2$ and with sectional curvatures within $[a, b]$ for $a \leq b < 0$. By [BGS85] M is the interior of a compact manifold with boundary which we denote by M_π .

A group G is said to *split over a subgroup* C if $G = A *_C B$ or $G = A *_C$ where $A \neq C \neq B$. It is well-known that A and B necessarily have infinite index in G . Note that G splits over C iff G act without edge inversions on a simplicial tree with no proper invariant subtree, no global fixed point, and exactly one orbit of edges such that C is a stabilizer of some edge [Ser80].

The purpose of this section is to prove that π does not split over a virtually nilpotent group. When M is a closed manifold, this can be shown simply by looking at the Mayer-Vietoris sequence of the splitting (see e.g. [Bel99]).

Noncompact case is more subtle. Main ingredients of the proof are again Mayer-Vietoris sequence, and the following splitting-theoretic lemma of B. Bowditch. More general (and harder to prove) results can be deduced from a recent paper of Bowditch [Bow98] who studies the structure of splittings of relatively hyperbolic groups over subgroups of peripheral groups. By contrast, our approach is elementary: we use basic manifold topology and Bass-Serre theory.

Lemma 3.1. *Suppose that π splits over a virtually nilpotent subgroup. Then π also splits over a virtually nilpotent subgroup C as $A *_C B$ or $A *_C$ where $A \neq C \neq B$ in such a way that a conjugate of any maximal parabolic subgroup lies in A or B . Furthermore, if C is parabolic, then the splitting can be chosen so that the maximal parabolic subgroup containing C lies in A .*

Proof. The proof is essentially borrowed from [Bow98, 3.5, 5.2] where a more general situation is considered. We specialize terminology to our case and give more details when seems appropriate.

Since π splits over a virtually nilpotent subgroup, π acts without edge inversions on a simplicial tree T with no proper invariant subtree, no global fixed point, and exactly one orbit of edges. We seek to construct a π -action on a (perhaps another) tree with the above properties such that every maximal parabolic subgroup of π fixes a vertex.

Fix an edge e of T and denote its stabilizer in π by π_e . Let P be the maximal virtually nilpotent subgroup of π that contains π_e . First, note that any maximal parabolic subgroup P' other than P fixes a vertex of T . (Indeed, every edge stabilizer of T is a conjugate of π_e , in particular, it lies in a conjugate of P , hence it must have trivial intersection with P' . So, unless P' fixes a vertex of T , we get that P' splits over the trivial group which is impossible because P' is noncyclic virtually nilpotent, and hence one-ended.) In particular, if P is not parabolic, then the π -action on T satisfies the desired properties.

Now assume that P fixes a vertex v of T . Then the fixed-point-set of π_e contains the segment joining v and a vertex of e . In particular, π_e lies in $\pi_{\bar{e}}$, the stabilizer of an edge \bar{e} adjacent to v . Since $\pi_{\bar{e}}$ is conjugate to π_e , it is virtually nilpotent. Hence $\pi_{\bar{e}}$ lies in a maximal parabolic subgroup containing π_e which is P . Thus the splitting of π over $\pi_{\bar{e}}$ satisfies the desired properties.

It remains to consider the case when P is parabolic and P does not fix a vertex of T . Let τ be the unique P -invariant minimal subtree of T . Let g be an element of π such that $g^{-1}e$ is an edge of τ (such a g exists since there is only one orbit of edges). Then the stabilizer of $g^{-1}e$ in π is $g^{-1}\pi_e g$. Note

that the group $g^{-1}\pi_e g \cap P$ (which is the stabilizer of $g^{-1}e$ in P) is infinite because otherwise P splits over a finite group $g^{-1}\pi_e g \cap P$ and P cannot since noncyclic nilpotent groups are one ended. Since $\pi_e \leq P$, we conclude that $g^{-1}Pg \cap P$ is infinite, therefore $P = g^{-1}Pg$ and, hence, $g \in P$ because any maximal parabolic subgroup is equal to its normalizer. The above argument has several implications as follows.

- e must be an edge of τ . (Indeed, using that there is only one orbit of edges we find $g \in \pi$ such that $g^{-1}e$ is an edge of τ . By above g must belong to P and since P stabilizes τ , e is an edge of τ .)
- P is equal to the setwise stabilizer of τ . (Indeed, if $h\tau = \tau$, then h takes the edge e of τ to an edge of τ . Hence, again $h \in P$.)
- Any edge of T lies in a unique π -image of τ . (Indeed, any edge of T is π -equivalent to e , so it is an edge of a tree that is a π -image of τ . Uniqueness of this tree is deduced as follows. Suppose, arguing by contradiction, that two trees that are in π -image of τ share an edge. Applying an element of π we can assume these two trees are τ and, say, $f\tau$ for some $f \in \pi$. This common edge is of the form $g^{-1}e$ for some $g \in \pi$, so by above $g \in P$. Hence, e is the common edge of $\tau = g\tau$ and $gf\tau$. So e is gf -image of an edge of τ , or equivalently, $f^{-1}g^{-1}e$ is an edge of τ . By the same argument, $gf \in P$, so $f \in P$ which contradicts to $\tau \neq f\tau$.)

Let Q be the set of π -images of τ . Construct now a graph $S(Q)$ with the vertex set $V(T) \cup Q$ where we deem two vertices v, η adjacent iff $v \in V(T)$, $\eta \in Q$, and $v \in \eta$.

First, prove that $S(Q)$ is a (simplicial) tree. Indeed, to show that $S(Q)$ is connected it suffices to find an arc that joins two arbitrary vertices $x, y \in V(T)$. The shortest arc that joins x and y in T can be uniquely written in the form $x_0x_1 \dots x_n$ where $x_0 = x$, $x_n = y$ and the arc $x_{i-1}x_i$ is contained in the tree $\tau_i \in Q$ (any edge of T lies in a unique tree from Q). Then the arc $x_0\tau_1x_1 \dots \tau_nx_n$ joins x and y in $S(Q)$. Second, show that $S(Q)$ has no circuits. Indeed, suppose $x_1\tau_1x_2 \dots \tau_nx_n$ is a circuit in $S(Q)$ with $x_1 = x_n$. Let α_i be the arc in τ_i that connects x_i to x_{i+1} . Then $\alpha_1 \cup \alpha_2 \dots \cup \alpha_n$ is a circuit in T , a contradiction.

The group π acts on $S(Q)$ without edge inversions. Any maximal parabolic subgroup now fixes a vertex. In particular, P is the stabilizer of τ . Since π is not virtually nilpotent, there exists a vertex v of τ whose stabilizer π_v is not a subgroup of P . (If the stabilizers of two adjacent vertex groups of τ were subgroups of P , then, since T has only one π -orbit of edges, the groups A and

B would lie in a conjugate of P . Hence $\pi = \langle A, B \rangle$ would lie in the conjugate of P .)

Now look at the edge d that joins vertices v and τ . The stabilizer π_d of d is a subgroup of P , so it is not equal to π_v . If $\pi_d = P$, then P has to stabilize v as well, hence P fixes a point of T which is not the case. Thus, π_d is not equal to the stabilizers of v and τ .

Now if any vertex of T is π -equivalent to v , then $S(Q)$ has one orbit of edges and $\pi \cong \pi_v *_{\pi_d} P$ is the desired splitting. Otherwise, T has two orbits of vertices represented by v and $v' \in \tau$ and hence $S(Q)$ has two orbits of edges d, d' . This defines a splitting $\pi \cong \pi_v *_{\pi_d} (P *_{\pi_{d'}} \pi_{v'})$ over π_d as needed. \square

3.2. Gluing aspherical cell complexes. In this section we frequently use the following standard construction. Let $f : X \rightarrow Y$ and $g : X \rightarrow Z$ be (cellular) maps of cell complexes that induce π_1 -injections on each connected component of X . Assume that Y, Z and each connected component of X are aspherical. Form a cell complex by gluing $X \times [0, 1]$ to Y and Z where $(x, 0)$ and $(x, 1)$ are identified with $f(x)$ and $g(x)$, respectively. The result is an aspherical cell complex (use Mayer-Vietoris in the universal covers and then the Hurewicz theorem). When X is connected, its fundamental group is isomorphic to $\pi_1(Y) *_{\pi_1(X)} \pi_1(Z)$. Similarly, if $Y = Z$ we can form a cell complex by gluing $X \times [0, 1]$ to Y where $(x, 0)$ and $(x, 1)$ are identified with $f(x)$ and $g(x)$, respectively. The result is, again, an aspherical cell complex and, if X is connected, its fundamental group is isomorphic to $\pi_1(Y) *_{\pi_1(X)}$.

3.3. Topological model for the splitting: amalgamated product case.

Assume that π splits over a virtually nilpotent group so that 3.1 gives an isomorphism $A *_C B \rightarrow \pi$ such that any maximal parabolic subgroup is conjugate to a subgroup of A or B , and the maximal virtually nilpotent subgroup P containing C lies in A . The inclusions of A and B into π define coverings $M_A \rightarrow M_\pi, M_B \rightarrow M_\pi$.

Note that for each connected component L of ∂M_π , the inclusion $L \hookrightarrow M_\pi$ lifts to M_A or M_B . (Indeed, maps into aspherical manifolds are homotopic iff the induced π_1 -homomorphisms are conjugate so $L \hookrightarrow M_\pi$ is homotopic to a map that takes $\pi_1(B)$ to A or B . This map lifts to the corresponding cover, and hence so does $L \hookrightarrow M_\pi$ by the covering homotopy theorem.) For every boundary component of M_π fix such a lift thereby defining an injection $\pi_0(\partial M_\pi) \rightarrow \pi_0(\partial M_A \cup \partial M_B)$. We refer to these components of $\partial M_A \cup \partial M_B$ as *lifted*.

Consider a closed aspherical manifold N_P with $\pi_1(N_P) \cong P$ defined as follows. If P is parabolic, N_P is the boundary component of M_π corresponding to the inclusion $P \hookrightarrow \pi$. If P is loxodromic, N_P is a circle embedded in M_π that realizes a generator of P . If P is trivial, N_P is a point of M_π .

In any case the inclusion $N_P \hookrightarrow M_\pi$ lifts to M_A , and we fix such a lift (in case P is parabolic, we use the same lift onto a boundary component of M_A that has been chosen above.) We use the notation \tilde{N}_P for the lifted manifold. The covering $N_C \rightarrow N_P$ induced by $C \hookrightarrow P$ lifts to a covering onto $\tilde{N}_P \subset M_A$.

Also fix a lift of $N_C \rightarrow N_P$ to M_B . This lift turns out to be a homeomorphism of N_C onto a boundary component of M_B because $C = A \cap B = P \cap B$. This component is denoted by \tilde{N}_C .

The above maps of N_C into M_A and M_B certainly become homotopic after projecting to M_π . Now we use this homotopy to build an aspherical cell complex Y by gluing M_A , M_B , and $N_C \times [0, 1]$ identifying $N_C \times \{0\}$ with \tilde{N}_P via the covering $N_C \rightarrow \tilde{N}_P$, and $N_C \times \{1\}$ with \tilde{N}_C via the chosen lift. Now we get a homotopy equivalence $Y \rightarrow M_\pi$ which extends the coverings $M_A \rightarrow M_\pi$ and $M_B \rightarrow M_\pi$.

Doubling M_π along the boundary produces a closed aspherical manifold DM_π . Similarly, the doubles of M_A , M_B along the lifted boundary components are denoted by DM_A , DM_B . Finally, let DY be the “double” of Y along the lifted boundary components, that is, DY is obtained by gluing DM_A , DM_B , and two copies of $N_C \times [0, 1]$ as above. The homotopy equivalence $Y \rightarrow M_\pi$ now extends to a homotopy equivalence $DY \rightarrow DM_\pi$.

It is useful to give another description of DY . Namely, let $\bar{D}M_B$ be the manifold obtained from DM_B by identifying two copies of $\tilde{N}_C \subset \partial DM_B$. Then DY is obtained by gluing DM_A , $\bar{D}M_B$, and $N_C \times [0, 1]$ where $N_C \times \{1\}$ is identified with $\tilde{N}_C \subset \bar{D}M_B$, and $N_C \times \{0\}$ is glued to $\tilde{N}_P \subset M_A$ via the covering $N_C \rightarrow \tilde{N}_P$.

3.4. Topological model for the splitting: HNN-extension case. As above, fix an isomorphism $A *_C \rightarrow \pi$ such that any maximal parabolic subgroup is conjugate to a subgroup of A or B , and the maximal virtually nilpotent subgroup P containing C lies in A . (We think of $A *_C$ as $\langle A, t | \phi(C) = t C t^{-1} \rangle$ where $\phi: C \rightarrow A$ is a monomorphism.)

We keep essentially the same notations as in the amalgamated product case. Thus the inclusion $A \hookrightarrow \pi$ defines a covering $M_A \rightarrow M_\pi$ and, we fix lifts of all boundary components of M_π to ∂M_A . Again, a lift of N_P to ∂M_A is denoted

by \tilde{N}_P . The covering $N_C \rightarrow N_P$ induced by the inclusion $C \hookrightarrow P$ of course lifts to a covering onto \tilde{N}_P using the chosen lift of N_P .

Also the composition $N_C \rightarrow N_P \hookrightarrow M_\pi$ can be lifted to a homeomorphism onto a component \tilde{N}_P of ∂M_A so that the lift induces $\phi: C \rightarrow A$. (Indeed, let f be the lift of $N_C \rightarrow N_P \hookrightarrow M_\pi$ to the universal covers such that f is equivariant with respect to the inclusion $C \hookrightarrow \pi$. Then $g = tf$ is a ϕ -equivariant homeomorphism onto a boundary component of the universal cover of M_π . So g descends to a covering of N_C onto a component of ∂M_A which is, in fact, a homeomorphism because $\phi(C)$ is maximal parabolic in A . The last statement is true since the maximal virtually nilpotent subgroup of π containing $\phi(C)$ is tPt^{-1} and $\phi(C) = tAt^{-1} \cap A = tPt^{-1} \cap A$.)

Again, $\bar{D}M_A$ is a double of M_A along the union of the lifted boundary components and \tilde{N}_C . As before we get a homotopy equivalence between DM_π and DY where DY is obtained by gluing $\bar{D}M_A$ and $N_C \times [0, 1]$ so that $N_C \times \{0\}$ is identified with a copy of \tilde{N}_C sitting inside $\bar{D}M_A$, while $N_C \times \{1\}$ is glued to \tilde{N}_P via the covering $N_C \rightarrow \tilde{N}_P$.

Theorem 3.5. *Let π be the fundamental group of a finite volume complete Riemannian manifold M of dimension $n > 2$ and with sectional curvatures within $[a, b]$ for $a \leq b < 0$. Then π does not split over a virtually nilpotent group.*

Proof. Assume first that the splitting of π given by 3.1 is $A *_C B$. Think of DY as glued from DM_A , $\bar{D}M_B$, and $N_C \times [0, 1]$. Since the splitting is nontrivial, both A and B have infinite index in π so M_A , M_B are noncompact. Hence DM_A , $\bar{D}M_B$ are noncompact since they are glued from noncompact spaces along compact subset. Now look at the Mayer-Vietoris sequence for homology with $\mathbb{Z}/2\mathbb{Z}$ -coefficients.

$$0 \rightarrow H_n(DY) \rightarrow H_{n-1}(N_C) \rightarrow H_{n-1}(DM_A) \oplus H_{n-1}(\bar{D}M_B) \rightarrow \dots$$

The map $H_{n-1}(N_C) \rightarrow H_{n-1}(DM_A) \oplus H_{n-1}(\bar{D}M_B)$ can be written as $i_{a*} \oplus -i_{b*}$ where $i_a: N_C \rightarrow DM_A$ is the covering onto \tilde{N}_P , and $i_b: N_C \rightarrow \bar{D}M_B$ is the homeomorphism onto \tilde{N}_C . Show that i_{b*} is injective. This is clear if the cohomological dimension of C is $< n - 1$. Otherwise, by the exact sequence of the pair $(\bar{D}M_B, \tilde{N}_C)$ it suffices to show that $H_n(\bar{D}M_B, \tilde{N}_C) = 0$ which is true by [Dol72, VIII.3.4]. (As stated, [Dol72, VIII.3.4] only applies when the complement of \tilde{N}_C in $\bar{D}M_B$ is connected. However, if $\bar{D}M_B \setminus \tilde{N}_C$ is nonconnected, it has two noncompact components each equal to M_B and the result

again follows since by the relative Mayer-Vietoris sequence $H_n(\bar{D}M_B, \tilde{N}_C)$ is a sum of two copies of $H_n(M_B, \tilde{N}_C) = 0$.)

By exactness $H_n(DY) = 0$ and we get a contradiction with the fact that DY is homotopy equivalent to the closed n -manifold DM_π .

Similarly, if $\pi \cong A *_C$, then we get the following Mayer-Vietoris sequence with $\mathbb{Z}/2\mathbb{Z}$ -coefficients [Bro82].

$$0 \rightarrow H_n(DY) \rightarrow H_{n-1}(N_C) \rightarrow H_{n-1}(\bar{D}M_A) \rightarrow \dots$$

The map $H_{n-1}(N_C) \rightarrow H_{n-1}(\bar{D}M_A)$ can be written as $i_* - f_*$ where $i: N_C \hookrightarrow \bar{D}M_A$ is the homeomorphism onto \tilde{N}_C , and f is the covering of N_C onto $\tilde{N}_P \hookrightarrow DM_A$. It remains to show that $i_* - f_*$ is injective which is clear if the cohomological dimension of C is $< n - 1$. Otherwise, look at the exact sequence of the pair $(\bar{D}M_A, \tilde{N}_C \amalg \tilde{N}_P)$ with $\mathbb{Z}/2\mathbb{Z}$ -coefficients. Again, by [Dol72, VIII.3.4] $H_n(\bar{D}M_A, \tilde{N}_C \amalg \tilde{N}_P) = 0$, hence the inclusion $\tilde{N}_C \amalg \tilde{N}_P \hookrightarrow \bar{D}M_A$ induces an injection on $(n - 1)$ -homology. Therefore,

$$(i_* - f_*)[N_C] = [N_C] - [N_P] \deg(N_C \rightarrow N_P)$$

is nonzero as wanted and we get a contradiction as before. \square

4 Convergence of finite volume manifolds

We refer to [Bel98a] or [Bel98b] for background. Here we only recall basic definitions and prove several new lemmas specific to the finite volume case. By an *action* of an abstract group π on a space X we mean a group homomorphism $\rho: \pi \rightarrow \text{Homeo}(X)$. An action ρ is called *free* if $\rho(\gamma)(x) \neq x$ for all $x \in X$ and all $\gamma \in \pi \setminus \text{id}$. In particular, if ρ is a free action, then ρ is injective.

4.1. Equivariant pointed Lipschitz topology. Let Γ_k be a discrete subgroup of the isometry group of a complete Riemannian manifold X_k and p_k be a point of X_k . The class of all such triples $\{(X_k, p_k, \Gamma_k)\}$ can be given the so-called equivariant pointed Lipschitz topology [Fuk86]; when Γ_k is trivial this reduces to the usual pointed Lipschitz topology.

If (X_k, p_k) is a sequence of simply-connected complete Riemannian n -manifolds with $a \leq \text{sec}(X_k) \leq 0$, then (X_k, p_k) subconverges in the pointed Lipschitz topology to (X, p) where X is a C^∞ -manifold with a complete $C^{1,\alpha}$ -Riemannian metric of Alexandrov curvature $\geq a$ and ≤ 0 . In fact, in a suitable harmonic atlas on X , the sequence (X_k, p_k) subconverges to (X, p) in pointed $C^{1,\alpha}$ topology [Fuk86, And90].

4.2. Pointwise convergence topology. Suppose that the sequence (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology. This allows one to talk about the *convergence* of a sequence of points $x_k \in X_k$ to $x \in X$. Furthermore, a sequence of isometries $\gamma_k \in \text{Isom}(X_k)$ we say that γ_k *converges*, if for any $x \in X$ and any sequence $x_k \in X_k$ that converges to x , $\gamma_k(x_k)$ converges. The limiting transformation γ that takes x to the limit of $\gamma_k(x_k)$ is necessarily an isometry of X .

Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be a sequence of isometric actions of a group π on X_k . We say that a sequence of actions (X_k, p_k, ρ_k) *converges in the pointwise convergence topology* if $\rho_k(\gamma)$ converges for every $\gamma \in \pi$. The map $\rho : \Gamma \rightarrow \text{Isom}(X)$ that takes γ to the limit of $\rho_k(\gamma)$ is necessarily a homomorphism.

Lemma 4.3. *Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be a sequence of free isometric actions of a discrete group π on Hadamard n -manifolds X_k of sectional curvatures within $[a, b]$ for $a \leq b < 0$. Assume that the sequence $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X, p, Γ) and (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology. If $\rho(\pi)$ has finite index in Γ , then $\Gamma = \rho(\pi)$.*

Proof. Being the fundamental group of a pinched negatively curved manifold, π is torsion free and any abelian subgroup of π is finitely generated [BS87, Bow93]. Therefore, every element in π is a power of a primitive element (an element of a group is called primitive if it is not a proper power). Fix an arbitrary $h \in \Gamma$ and find $g \in \pi$ so that $h^m = \rho(g)$ for some $m \geq 1$. By passing to the appropriate root we can assume g is primitive. There always exists a sequence $\rho_k(g_k)$ that converges to h . Then both $\rho_k(g_k^m)$ and $\rho_k(g)$ converge to h^m . Hence $g_k^m = g$ for large k by [Bel98a, lem 2.6]. Since g is primitive, $m = 1$ so $h \in \rho(\pi)$ as wanted. \square

In general, if $|sec|$ is bounded above, the injectivity radius is at best upper-semicontinuous. However, it becomes continuous if the curvature is also non-positive.

Lemma 4.4. *If (M_k, p_k) be a sequence of pointed complete Riemannian manifolds of sectional curvatures within $[a, 0]$ for $a \leq 0$. Assume that (M_k, p_k) converges to (M, p) in pointed $C^{1,\alpha}$ topology. Then for any $x_k \in M_k$ that converges to $x \in M$, the injectivity radii of M_k at x_k converge to the injectivity radius of M at x .*

Proof. Arguing by contradiction find x_k that converges to x while $\text{inj}(x_k)$ does not approach $\text{inj}(x)$. Pass to a subsequence so that $\text{inj}(x_k)$ converges to $c \neq \text{inj}(x)$. Write $M_k = X_k/\Gamma_k$ and $M = X/\Gamma$ and pick preimages \tilde{x}_k of x_k , and \tilde{x} of x . Since $|\text{sec}(M_k)|$ is uniformly bounded, a result in [Fuk86] implies that $(X_k, \tilde{x}_k, \Gamma_k)$ subconverges to (X, \tilde{x}, Γ) in the equivariant pointed Lipschitz topology. Now since the sectional curvature is nonpositive, $\text{inj}(x) = d_\Gamma(\tilde{x})/2$ and $\text{inj}(x_k) = d_{\Gamma_k}(\tilde{x}_k)/2$ where $d_\Gamma(x)$ is the infimal displacement of the point x by the isometries of Γ . It follows easily from the definition of equivariant Lipschitz convergence that $d_{\Gamma_k}(\tilde{x}_k)$ converges to $d_\Gamma(x)$ so we get a contradiction. \square

4.5. Nikolaev's smoothing theorem. A useful technical tool is the following smoothing theorem of I. Nikolaev [Nik91]. We only state an easy case of $C^{1,\alpha}$ metrics even though [Nik91] actually applies to any path metric. Let (M, g) be a complete $C^{1,\alpha}$ -Riemannian manifold of Alexandrov curvature $\geq a$ and $\leq b$. Then there exist a sequence of complete Riemannian metrics g_m on M with sectional curvatures within $[a - 1/m, b + 1/m]$ such that (M, g_m) converges to (M, g) in (unpointed!) $C^{1,\alpha}$ -topology. Also $\text{id} : (M, g_m) \rightarrow (M, g)$ and $\text{id} : (M, g) \rightarrow (M, g_m)$ are $2^{1/m}$ -Lipschitz.

Proposition 4.6. *Let X_k be a sequence of Hadamard n -manifolds with sectional curvatures in $[a, b]$ for $a \leq b < 0$ and $n \geq 3$. Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be an arbitrary sequence of free and isometric actions such that each $X_k/\rho_k(\pi)$ has finite volume. Then after passing to a subsequence, there are points $p_k \in X_k$ such that*

- (i) (X_k, p_k, ρ_k) converges to in the pointwise convergence topology to a free action (X, p, ρ) , and $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to $(X, p, \rho(\pi))$;
- (ii) If q_k, q are projections of p_k, p , respectively, then $(X_k/\rho_k(\pi), q_k)$ converges to $(X/\rho(\pi), q)$ in the pointed $C^{1,\alpha}$ topology;
- (iii) the $C^{1,\alpha}$ -manifold $X/\rho(\pi)$ has finite volume, and is diffeomorphic to the interior of a compact manifold with boundary; the number of connected components of the boundary is the number of maximal parabolic subgroups of π .

Proof. Let $M_k = X_k/\rho_k(\pi)$. By 3.5 and [Bel98a, 2.7, 2.10] we can assume by passing to a subsequence that (X_k, p_k, ρ_k) converges to (X, p, ρ) in pointwise

convergence topology and $(X_k, p_k, \rho_k(\pi))$ converges to (X, p, Γ) in equivariant pointed Lipschitz topology. By [Bel98a, 2.5] we know that the Γ -action on X is free. By [Fuk86, And90] (M_k, q_k) converges to $(X/\Gamma, q)$ in the pointed $C^{1,\alpha}$ topology.

Now $M = X/\rho(\pi)$ is a smooth manifold with a $C^{1,\alpha}$ Riemannian metric g of Alexandrov curvature of g is within $[a, b]$. Let (M, g_m) be the Nikolaev's smoothing of (M, g) [Nik91]. Since (M, g_m) is a complete Riemannian manifold of pinched negative curvature which is homotopy equivalent to M_k , [Sch84] implies that (M, g_m) has finite volume so [BGS85] implies that M is the interior of a compact manifold with boundary and the number of boundary components is the same as the number of ends of M_k (or, alternatively, the number of maximal parabolic subgroups of π).

Since (M, g_m) converge to (M, g) in unpointed $C^{1,\alpha}$ -topology, $\text{vol}(M, g_m)$ tends to $\text{vol}(M, g)$. Also $\text{id} : (M, g_m) \rightarrow (M, g_{m+1})$ is Lipschitz with Lipschitz constant approaching 1, hence the volumes of (M, g_m) are uniformly bounded [BGS85, p11]. Thus the volume of (M, g) is finite. Being a quotient of M , the manifold X/Γ also has finite volume so $\rho(\pi)$ has finite index in Γ . Hence $\rho(\pi) = \Gamma$ by 4.3. \square

Remark 4.7. Another obvious application of Nikolaev's smoothing and the continuity of injectivity radius is that

$$\{\text{inj}_{g_k} \leq \epsilon 2^{-1/k}\} \subset \{\text{inj}_g \leq \epsilon\} \subset \{\text{inj}_{g_k} \leq \epsilon 2^{1/k}\}.$$

In particular, ϵ -thick part $\{\text{inj}_g \geq \epsilon\}$ is compact since it is so for C^∞ metrics.

Lemma 4.8. *The injectivity radius and the diameter of ϵ -thick part are bounded above and below on $\mathcal{M}_{a,b,\pi,n}$.*

Proof. Arguing by contradiction, let $M_k \in \mathcal{M}_{a,b,\pi,n}$ be a sequence of manifolds with the diameter of the ϵ -thick part going to infinity for some ϵ . Find points $p_k \in M_k$ such that (M_k, p_k) subconverges to (M, p) . By 4.7 the ϵ -thick part of M is compact hence it lies in the open ball $B(p, R)$ for some R . Pass to subsequence so that diameters of the ϵ -thick parts of M_k are $> 2R+2$; thus, for all large k , the ϵ -thick part of M_k contains a point that lies in $M_k \setminus B(p_k, R+1)$. Furthermore, by continuity of injectivity radius $B(p_k, R+1)$ contains a point of injectivity radius $\geq \epsilon$ for all large k . Since $n > 2$, the ϵ -thick part is connected, so for all large k , $B(p_k, 2R+1) \setminus B(p_k, R+1)$ contains a point of injectivity radius $\geq \epsilon$. This point subconverges to a point in $B(p, 2R+2) \setminus B(p, R+1/2)$ of injectivity radius $\geq \epsilon$ which is a contradiction.

Now a lower bound on the injectivity radius (and hence on the diameter of the ϵ -thick part) is provided by the Margulis lemma [BGS85]. As for an upper bound assume there is a sequence of manifolds (M_k, p_k) in with $M_k \in \mathcal{M}_{a,b,\pi,n}$ and $\text{inj}_{p_k} > k$. Pass to a subsequence so that (M_k, p_k) converges to (M, p) . Since the ϵ -thick part of M is compact, the injectivity radius of M is bounded above which contradicts the continuity of the injectivity radius. \square

Lemma 4.9. *There exist $\epsilon > 0$ such that for any $M \in \mathcal{M}_{a,b,\pi,n}$ the ϵ -thin part of M consists only of cusps.*

Proof. Arguing by contradiction find a sequence of manifolds $M_k = X_k/\rho_k(\pi) \in \mathcal{M}_{a,b,\pi,n}$ each containing a closed geodesic of length $< 1/k$. By 4.6 and [Fuk86] M_k subconverges in the pointed Lipschitz topology. Denote the limiting manifold by $(M, g) = X/\rho(\pi)$.

By 4.7 the ϵ -thick part of (M, g) contains the 2ϵ -thick part of (M, g_1) . For large k Lipschitz approximations ϕ_k between M_k and M take the ϵ -thick part of M_k Hausdorff close to the ϵ -thick part of M . Hence for all large k the 2ϵ -thick part of (M, g_1) is contained in the ϕ_k -image of the ϵ -thick part of M_k .

Now let ϵ be so small that 2ϵ -thin part of (M, g_1) consists only of cusps. Thus, the ϕ_k -image of any ϵ -tube of M_k lies in a 2ϵ -cusp of (M, g_1) . Hence if γ_k be a closed geodesic in a tube of M_k , then $\phi_k(\gamma_k)$ represents a parabolic element. By algebraic reasons, $n > 2$ implies that $\rho_k \circ \rho^{-1}$ takes parabolics to parabolics. So γ_k represents a parabolic element which is a contradiction. \square

5 Group theoretic applications

In this section π is a group such that $\mathcal{M}_{a,b,\pi,n}$ is nonempty.

Corollary 5.1. *$\text{Out}(\pi)$ is finite.*

Proof. Let $\rho_k \in \text{Aut}(\pi)$ lie in different conjugacy classes. This defines a sequence ρ_k of free isometric actions of π on the universal cover X of M . For a finite generating set S of π , let $D_k(x) = \max_{\gamma \in S} \{d(x, \rho_k(\gamma)(x))\}$ and $D_k = \inf\{D_k(x) : x \in X\}$. As in [Bel98a, 2.10], we can choose a sequence of points $x_k \in X$ such that $D_k(x_k) \leq D_k + 1/k$. If $D_k \rightarrow \infty$, we get an action on an \mathbf{R} -tree hence a splitting which is impossible. So assume $D_k(x_k)$ is uniformly bounded.

Let F be the Dirichlet fundamental domain for the action of π on X . There exists a sequence $\phi_k \in \pi$ such that $\phi_k(x_k) \in F$. Then $(X, \phi_k(x_k), \phi_k \rho_k \phi_k^{-1})$ subconverges in the pointwise convergence topology.

Suppose first that $\phi_k(x_k)$ is precompact, so that passing to subsequence we can assume that $\phi_k(x_k)$ converges to $x \in X$. Then $(X, x, \phi_k \rho_k \phi_k^{-1})$ also subconverges in the pointwise convergence topology, in other words, passing to subsequence, we deduce that $\phi_k \rho_k(\gamma) \phi_k^{-1}$ converges in $\text{Isom}(X)$ for any $\gamma \in \pi$. Since π is a closed subgroup the limit lies in π . So $\phi_k \rho_k \phi_k^{-1}$ converges in $\text{Hom}(\pi, \pi)$. Since the space $\text{Hom}(\pi, \pi)$ is discrete, $\phi_k \rho_k \phi_k^{-1}$ are all equal for large k . So ρ_k lie in the same conjugacy class for large k , a contradiction.

If $\phi_k(x_k)$ is not precompact, then passing to subsequence we can assume $\phi_k(x_k)$ converges a parabolic fixed point. (The closure of F at infinity is just finitely many parabolic fixed points.) Choose a π -invariant set of mutually disjoint horoballs. Passing to subsequence, we assume that $\phi_k(x_k)$ lies in one horoball H for all k . Since π is not virtually nilpotent, for each k there is a generator $\gamma_k \in S$ such that the horoballs H and $\phi_k \rho_k \phi_k^{-1}(\gamma_k)(H)$ are disjoint. As S is finite we can pass to subsequence so that H and $\phi_k \rho_k \phi_k^{-1}(\gamma)(H)$ are disjoint for all k and some $\gamma \in S$. Since $D_k(x_k)$ is uniformly bounded, the distance between $\phi_k(x_k)$ and $\phi_k \rho_k(\gamma) \phi_k^{-1}(\phi_k(x_k))$ is uniformly bounded. On the other hand, this distance has to converge to infinity because it is bounded below by the distance from $\phi_k(x_k)$ to the horosphere ∂H . This contradiction completes the proof. \square

Corollary 5.2. *π is cohopfian.*

Proof. Arguing by contradiction assume there exists an injective homomorphism $\phi : \pi \rightarrow \pi$ which is not onto. By [Sch84] the manifold $X/\phi(\pi)$ has finite volume hence $\phi(\pi)$ must be of finite index, say $m > 1$, in π . Iterating ϕ , we get a sequence of free isometric actions ρ_k of π on the universal cover X of M such that $\rho_{k+1}(\pi)$ is an index m subgroup of $\rho_k(\pi)$ for each k .

Same proof as before gives that $\phi_k \rho_k \phi_k^{-1}$ are all equal for large k . In particular, $X/\rho_k(\pi)$ and $X/\rho_{k+1}(\pi)$ are isometric for all large k . But there exists an m -sheeted cover $X/\rho_{k+1}(\pi) \rightarrow X/\rho_k(\pi)$, hence $\text{vol}(X/\rho_{k+1}(\pi)) = m \cdot \text{vol}(X/\rho_k(\pi))$. Thus, $m = 1$, a contradiction. \square

Remark 5.3. There is of course another proof that π is cohopfian. Namely, by [Sch84] $\phi(\pi)$ has finite index m in π . Since X/π and $X/\phi(\pi)$ are properly homotopy equivalent, they have equal simplicial volumes (which are also

nonzero [Gro82]). On the other hand, simplicial volume is multiplicative under finite covers so $m = 1$.

6 Diffeomorphism finiteness

In this section we prove a theorem that implies 1.1 when combined with 4.6. Main ingredients of the proof are exponential convergence of geodesics and continuity of injectivity radius. We also use in a crucial way that the metrics converge in at least C^1 topology.

First, we need a better understanding of cusps for manifolds in $\mathcal{M}_{a,b,\pi,n}$. Fix an $\epsilon \in (0, \mu_{n,a})$ where $\mu_{n,a}$ is the Margulis constant and fix an ϵ -cusp of a manifold $M = X/\pi \in \mathcal{M}_{a,b,\pi,n}$. Denote by inj_ϵ the boundary of the cusp; thus inj_ϵ is a compact topological submanifold of M of codimension one. Set $d_\epsilon = \text{diam}(inj_\epsilon)$. Also let H_ϵ^+ , H_ϵ^- be the quotients of horospheres such that inj_ϵ lies in the region bounded by H_ϵ^+ and H_ϵ^- , and $H_\epsilon^+ \cap inj_\epsilon$, $H_\epsilon^- \cap inj_\epsilon$ are nonempty. We also assume that H_ϵ^- is “closer” to infinity than H_ϵ^+ (i.e. $\text{diam}(H_\epsilon^-) \leq \text{diam}(H_\epsilon^+)$). Following 2.4 we let $a = -K^2$, $b = -k^2$.

Lemma 6.1. *Given ϵ, σ satisfying $0 < \sigma < \epsilon < \min\{\mu_{n,a}, 1\}$, fix an ϵ -cusp of a manifold $M = X/\pi \in \mathcal{M}_{a,b,\pi,n}$. Then the following holds:*

- (1) $\text{diam}(H_\epsilon^+) \leq 3d_\epsilon$;
- (2) $\text{diam}(inj_\sigma) \leq 2d_\epsilon + k^{-1} \ln\left(\frac{2\epsilon}{\sigma}\right) - K^{-1} \ln\left(\frac{\epsilon}{2\sigma}\right)$
- (3) $K^{-1} \ln\left(\frac{\epsilon}{2\sigma}\right) - d_\epsilon \leq \text{dist}(inj_\epsilon, inj_\sigma) \leq k^{-1} \ln\left(\frac{2\epsilon}{\sigma}\right) + 2d_\epsilon$;
- (4) if $\sigma < \frac{\epsilon}{2} e^{-Kd_\epsilon}$, then $\text{diam}(H_\sigma^+) \leq \text{diam}(H_\epsilon^-)$.

Proof. To prove (1) find $x, y \in H_\epsilon^+$ with $\text{dist}(x, y) = \text{diam}(H_\epsilon^+)$. Let $\tilde{x}, \tilde{y} \in inj_\epsilon$ be the points obtained by pushing x, y along radial geodesics. Then by the triangle inequality $\text{dist}(x, y) \leq \text{dist}(x, \tilde{x}) + \text{dist}(\tilde{x}, \tilde{y}) + \text{dist}(\tilde{y}, y) \leq 3d_\epsilon$ where the latter inequality holds because Busemann functions are 1-Lipschitz.

It remains to prove (2) – (4). Let z be a point of the ideal boundary of X corresponding to the cusp under consideration. We denote by \widetilde{inj}_δ , \widetilde{H}_δ^+ , \widetilde{H}_δ^- the lifts of inj_δ , H_δ^+ , H_δ^- to the universal cover which are in bounded distance from a horosphere about z . Let $\gamma(t)$ be a geodesic asymptotic to z with $\gamma(0) \in \widetilde{H}_\epsilon^+$ and assume that $\gamma(t)$ intersects \widetilde{inj}_σ and \widetilde{inj}_ϵ in the points $\gamma(t_\sigma)$ and $\gamma(t_\epsilon)$, respectively. Note that $t_\sigma > t_\epsilon$ and $t_\epsilon \in [0, d_\epsilon]$.

Since $\text{inj}(\gamma(t_\sigma)) = \sigma$, one can find $g \in \pi$ such that $d(g(\gamma(t_\sigma)), \gamma(t_\sigma)) = 2\sigma$. Now $d(g(\gamma(t_\epsilon)), \gamma(t_\epsilon)) \geq 2\epsilon$. Thus, by 2.4

$$\frac{\epsilon}{\sigma} = \frac{2\epsilon}{2\sigma} \leq \frac{d(g(\gamma(t_\epsilon)), \gamma(t_\epsilon))}{d(g(\gamma(t_\sigma)), \gamma(t_\sigma))} \leq 2e^{K(t_\sigma - t_\epsilon)} \quad \text{or} \quad K^{-1} \ln \left(\frac{\epsilon}{2\sigma} \right) \leq t_\sigma - t_\epsilon.$$

Similarly, since $\text{inj}(x_\epsilon) = \epsilon$, one can find $h \in \pi$ with $d(h(\gamma(t_\epsilon)), \gamma(t_\epsilon)) = 2\epsilon$. Again, $2\sigma \leq d(h(\gamma(t_\sigma)), \gamma(t_\sigma))$ so

$$\frac{\epsilon}{\sigma} \geq \frac{d(h(\gamma(t_\epsilon)), \gamma(t_\epsilon))}{d(h(\gamma(t_\sigma)), \gamma(t_\sigma))} \geq e^{k(t_\sigma - t_\epsilon)/2} \quad \text{or} \quad k^{-1} \ln \left(\frac{2\epsilon}{\sigma} \right) \geq t_\sigma - t_\epsilon.$$

So $t_\sigma \in [K^{-1} \ln \left(\frac{\epsilon}{2\sigma} \right), d_\epsilon + k^{-1} \ln \left(\frac{2\epsilon}{\sigma} \right)]$. Hence, by the triangle inequality

$$\text{diam}(\text{inj}_\sigma) \leq 2d_\epsilon + k^{-1} \ln \left(\frac{2\epsilon}{\sigma} \right) - K^{-1} \ln \left(\frac{\epsilon}{2\sigma} \right) \quad \text{and}$$

$$K^{-1} \ln \left(\frac{\epsilon}{2\sigma} \right) - d_\epsilon \leq \text{dist}(\text{inj}_\epsilon, \text{inj}_\sigma) \leq 2d_\epsilon + k^{-1} \ln \left(\frac{2\epsilon}{\sigma} \right).$$

Furthermore, if $K^{-1} \ln \left(\frac{\epsilon}{2\sigma} \right) > d_\epsilon$, then H_σ^+ is “closer” to infinity than H_ϵ^- as desired. \square

Corollary 6.2. *The volume function is uniformly bounded above on $\mathcal{M}_{a,b,\pi,n}$.*

Proof. First, show that the volume of the σ -thick part is uniformly bounded above on $\mathcal{M}_{a,b,\pi,n}$ for any $\sigma \in (0, \mu_{a,n})$. Indeed, observe that the diameter of $M_{[\sigma,\infty)}$ is bounded above by 4.8. Hence, $M_{[\sigma,\infty)}$ is in the image of a ball in X of some uniformly bounded above radius. By Bishop-Gromov volume comparison the volume of the ball is uniformly bounded above and the result follows because the projection $X \rightarrow M$ is volume non-increasing.

We fix an $\epsilon \in (0, \mu_{a,n})$, and an arbitrary ϵ -cusp of $M \in \mathcal{M}_{a,b,\pi,n}$. Now we seek to obtain a uniform upper bound on $\text{vol}(H_\epsilon^-)$. Let $\sigma = \frac{\epsilon}{2}e^{-K(d_\epsilon+1)}$ so that H_σ^+ is “closer” to infinity than H_ϵ^- . Let T be the distance between H_σ^+ and H_ϵ^- ; note that $T \in [d_\epsilon + 1, (d_\epsilon + 1)K/k + (\ln 4)/k]$. By above, the volume enclosed between inj_ϵ and inj_σ is bounded above on $\mathcal{M}_{a,b,\pi,n}$ by a constant V depending only on σ, a, b, π, n . The same is then true for the volume enclosed between H_σ^+ and H_ϵ^- which is equal to $\int_0^T \text{vol}(H_t)dt$ where H_t is the quotient of a horosphere at t -level, and $H_0 = H_\epsilon^-$, $H_T = H_\sigma^+$.

By 2.4, pushing along radial vector field gives an e^{Kt} -Lipschitz map $H_t \rightarrow H_0$. Thus, $\text{vol}(H_0) \leq \text{vol}(H_t)e^{Knt}$ (in this proof we always equip H_t with the Riemannian metric induced by the inclusion into M). Hence by the Fubini's

theorem (which applies since the Busemann function is a C^2 -Riemannian submersion) we have

$$\text{vol}(H_0)T = \int_0^T \text{vol}(H_0)dt \leq e^{KnT} \int_0^T \text{vol}(H_t)dt \leq e^{KnT}V.$$

Thus $\text{vol}(H_0) = \text{vol}(H_\epsilon^-)$ is uniformly bounded above over $\mathcal{M}_{a,b,\pi,n}$.

Now pushing along radial vector field gives an e^{-kt} -Lipschitz diffeomorphism $H_0 \rightarrow H_t$ so $\text{vol}(H_t) \leq \text{vol}(H_0)e^{-ktn}$. Hence

$$\int_0^\infty \text{vol}(H_t)dt \leq \text{vol}(H_0) \int_0^\infty e^{-ktn}dt \leq \text{vol}(H_0)/kn$$

and we get a uniform upper bound for the volume of the σ -cusp under consideration. Thus, we get a uniform upper bound on the volume of M . \square

Theorem 6.3. *Let (M_k, g_k) be a sequence of manifolds in $\mathcal{M}_{a,b,\pi,n}$ such that for some $p_k \in M_k$, (M_k, p_k) converges in the pointed $C^{1,\alpha}$ -topology to (M, p) equipped with a $C^{1,\alpha}$ Riemannian metric g . Then for all large k there exist diffeomorphisms $f_k: M \rightarrow M_k$ such that the pullback metrics $f_k^*g_k$ converge to g in $C^{1,\alpha}$ topology uniformly on compact subsets.*

Proof. Renumerate the sequence so that M_k now come with even indices while odd indices correspond to Nikolaev's smoothings of (M, g) . We still denote the obtained sequence by M_k . Since (M_k, p_k) converges to (M, p) in the pointed $C^{1,\alpha}$ -topology there are $1/k$ -Lipschitz smooth embeddings $\phi_k: B_{1/k}(p_k) \rightarrow M$ with $d(p, \phi_k(p_k)) \leq 1/k$ such that ϕ_k -pushforward of g_k converges to g in $C^{1,\alpha}$ -topology uniformly on compact subsets. By [Nik91] we can take $\phi_{2k+1} = \text{id}_M$.

As we shall prove below, for each small enough ϵ and large enough k , there exists a diffeomorphism $h_{k,\epsilon}: M_k \rightarrow M$ that is equal to ϕ_k when restricted to the ϵ -thick part of M_k . Note that the continuity of injectivity radius implies that, given ϵ , the maps ϕ_k are defined on the 1-neighborhood of the ϵ -thick part of M_k for all $k \geq C(\epsilon)$ and some positive integer valued function C . Now the diffeomorphism $f_k = h_{k+C(1/k), 1/k}$ enjoys the desired properties. Thus, it remains to construct $h_{k,\epsilon}$.

Use 4.9 to find an ϵ' such that ϵ' -thin part of each M_k consists of cusps; we assume $\epsilon \in (0, \max\{1, \epsilon'/10\})$. Fix a cusp of M and the corresponding cusps of M_k . Using 6.1, we can make so ϵ is so small that

- H_{k,ϵ^2}^+ is closer to infinity than $H_{k,\epsilon}^-$, and $\text{dist}(H_{k,\epsilon^2}^+, H_{k,\epsilon}^-) > 10$;
- H_{k,ϵ^3}^+ is closer to infinity than H_{k,ϵ^2}^- , and $\text{dist}(H_{k,\epsilon^3}^+, H_{k,\epsilon^2}^-) > 10$.

(Here subindex k indicates that the quotients of the horospheres lie in a cusp of M_k .) Let R_k be the radial vector field defined on the ϵ -thin part of M_k , and let D_k be the 1-neighborhood of the region between inj_{k,ϵ^2} and inj_{k,ϵ^3} .

Using exponential convergence of geodesics, one can easily see that for any $\alpha \in (0, \pi/2)$ there exists r , depending only on α, a, b, n and independent of k , such that for any $x \in D_k$ and any y lying in the same ϵ^2 -cusp of M_k with $d_k(x, y) \geq r$, the angle at x formed by R_k and the tangent vector to the geodesic segment $[y, x]$ is $\leq \alpha$. Fix $\alpha = \pi/3$ and fix the corresponding r .

Assume k is large enough so that the embeddings ϕ_k are defined on the $2r$ -neighborhood of D_k . By continuity of injectivity radius the domains $\phi_k(D_k) \subset M$ converge to some compact set in Hausdorff topology and we can find a smooth domain D which is Hausdorff close to the set and satisfies $D \subset \phi_k(D_k)$ for all large k . Now use $\psi_k = \phi_k^{-1}$ to pullback all metrics to D .

We want to show that the region bounded by $\phi_k(H_{k,\epsilon^2}^-)$ and $\phi_{k+1}(H_{k+1,\epsilon^3}^+)$ is diffeomorphic to $H_{k,\epsilon^2}^- \times [0, 1]$. It suffices to produce a C^∞ nowhere vanishing vector field on D that is transverse to both $\phi_k(H_{k,\epsilon^2}^-)$ and $\phi_{k+1}(H_{k+1,\epsilon^3}^+)$. We shall construct such a vector field as a controlled approximation of $\psi_k^\# R_k$, the ψ_k -pullback of the radial vector field R_k . Choose a harmonic atlas $\{B_j\}$ on D as in [And90] in which the ψ_k -pullbacks of g_k converge to g in $C^{1,\alpha}$ -topology. Fix a partition of unity associated with the atlas. Let $y \in M$ be a point in the same cusp and $d(y, D) \geq r + 1$.

Now we construct a C^∞ vector field X_k on D by defining it on each chart neighborhood B_j and then gluing via the partition of unity. Look at the $2r$ -neighborhood of D equipped with the metric $\psi_k^\# g_k$. The preimage of B_j under $\exp_{\psi_k^\# g_k} : T_y M \rightarrow M$ is the disjoint union of copies of B_j . Pick a copy closest to the origin, join each of its points to the origin by rays, and then project the rays to M via $\exp_{\psi_k^\# g_k}$. Now the tangent vectors at the endpoints of the obtained geodesic segments joining y with points of B_j form a vector field on B_j . Gluing these local data via the partition of unity gives X_k . Note that by construction X_k is a nowhere vanishing vector field such that the angle formed by X_k and the exterior normal $\psi_k^\# R_k$ to $\phi_k(H_{k,\epsilon^2}^-)$ or $\phi_k(H_{k,\epsilon^3}^+)$ is within $(0, \pi/3]$.

Now on each chart X_k is a solution of the geodesic equation. Since metrics converge in $C^{1,\alpha}$ topology, Christoffel symbols converge in at least C^0 topology, so by standard ODE results [Rei71, I.5.8] X_k converges in C^0 -topology to some C^0 vector field X . So the angle, measured in the metric $\psi_k^\# g_k$, formed by X_m and $\psi_k^\# R_k$ is within $(0, c]$ for all m, k large enough, and some $c \in [\pi/3, \pi/2)$.

Now a standard differential topology arguments implies that the region between $\phi_k(H_{k,\epsilon^2}^-)$ and $\phi_{k+1}(H_{k+1,\epsilon^3}^+)$ is diffeomorphic to $H_{k,\epsilon^2}^- \times [0, 1]$ as needed.

Finally, we are ready to define $h_{k,\epsilon}$. Let $M_{k,\delta}$ be the compact manifold obtained from M_k by chopping off cusps along all surfaces $H_{k,\delta}^-$; we think of $M_{k,\delta}$ as a bounded domain in M_k . Define $h_{2k+1,\epsilon} = \text{id}_M$, and define $h_{2k,\epsilon}$ as the following composition. First, map M_{2k} diffeomorphically to the interior of M_{2k,ϵ^2} by a map which is the identity on the 1-neighborhood of $M_{2k,\epsilon}$. Then map M_{2k,ϵ^2} to M by ϕ_{2k} . Next use the above argument to map $\phi_{2k}(M_{2k,\epsilon^2})$ diffeomorphically onto $\phi_{2k+1}(M_{2k+1,\epsilon^3}) = M_{2k+1,\epsilon^3}$ by a map which is the identity on the 1-neighborhood of $M_{2k+1,\epsilon}$. Last, map M_{2k+1,ϵ^3} diffeomorphically to $M_{2k+1} = M$, again keeping $M_{2k+1,\epsilon}$ fixed. \square

7 Pinching

In this section we prove several pinching results that follow from 1.1.

Corollary 7.1. *Given a group π , there exists $\epsilon = \epsilon(\pi) > 0$ such that any finite volume manifold from $\mathcal{M}_{n,-1-\epsilon,-1,\pi}$ is diffeomorphic to a real hyperbolic manifold.*

Proof. Arguing by contradiction find a sequence M_k of manifolds with fundamental group isomorphic to π and sectional curvatures within $[-1 - 1/k, -1]$. By the main theorem, we can assume that (M_k, g_k) converges in $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ -Riemannian manifold (M, g) . Now the universal cover X of M is a complete $C^{1,\alpha}$ -Riemannian manifold of Alexandrov curvature -1 . By [Ale57] X is isometric to the real hyperbolic space and we are done. (Alternatively, one can repeat the argument below appealing to [Gao90] to deduce that g is a C^∞ metric.) \square

Remark 7.2. It follows from [Gro82] that ϵ in 7.1 only depends on the simplicial volume of M when $n > 3$. This is formally stronger than dependence on $\pi_1(M)$, yet it actually amounts to the same thing because the bounds $\|M\| < V$, $a \leq \text{sec}(M) \leq b < 0$ imply finitely many possibilities for $\pi_1(M)$ [Fuk84]. Furthermore, if n is even and > 3 , then ϵ in 7.1 only depends on the Euler characteristic $\chi(M)$. Indeed, it is well known that there is a positive constant C_n such that if $-1 - C_n \leq \text{sec}(M) \leq -1$, then the Gauss-Bonnet integrand χ satisfies $C_{n,1}\omega \leq \chi \leq C_{n,2}\omega$ for some constants $C_{n,1}, C_{n,2}$ and the Riemannian volume form ω . Now the Gauss-Bonnet formula (generalized to finite volume case in [CG85]) shows that bounds on $\text{vol}(M)$ and $\chi(M)$ are equivalent.

Remark 7.3. Note that if $n = 4$, then the Gauss-Bonnet formula (again see [CG85] in noncompact case) gives a stronger version of 1.1. Namely, the class of finite volume Riemannian 4-manifolds with sectional curvatures within $[a, b]$ where $a \leq b < 0$ and with uniformly bounded Euler characteristics is precompact in the $C^{1,\alpha}$ -topology. Indeed, it was shown by Milnor [Che55] that in dimension four the Gauss-Bonnet integrand χ satisfies $\chi \geq 3b^2\omega$ where ω is the volume form and $b < 0$ is the upper curvature bound. Hence the Gauss-Bonnet formula implies that, for any closed Riemannian 4-manifold M with sectional curvatures within $[a, b]$ where $b < 0$, volume is bounded by the Euler characteristic: $\chi(M) \geq 3b^2\text{vol}(M)$.

Berger's classification of holonomy groups implies the following possibilities for the restricted holonomy group of a complete negatively curved n -manifold: $\mathbf{SO}(n)$ (generic case), $\mathbf{U}(n/2)$ (Kähler), $\mathbf{Sp}(1)\mathbf{Sp}(n/4)$ (quaternionic-Kähler), and $\mathbf{Spin}(9)$ (Cayley). Any complete negatively curved manifold with restricted holonomy group $\mathbf{Spin}(9)$ is a quotient of the Cayley hyperbolic plane [Bes88, 10.96.VI]. The following uniformization theorem is due to Yeung [Yeu95]. Let M be a complete finite volume Riemannian manifold that is either Kähler or quaternionic-Kähler. In case M is noncompact assume also that sectional curvatures of M are within $[a, b]$ for some $a \leq b < 0$ with $a/b \leq (n-1)^4$. If M is homotopy equivalent to a complete pointwise quarter-pinched negatively curved Riemannian manifold, then M is locally symmetric.

Combining 1.1, the Yeung's theorem, some results of Gao [Gao90], and some linear algebra of Kähler curvature tensor, we get the following

Corollary 7.4. *Given a group π , there exists $\epsilon = \epsilon(\pi) > 0$ such that*

- (1) *any finite volume Kähler manifold from $\mathcal{M}_{n,-4-\epsilon,-1,\pi}$ is diffeomorphic to a scalar multiple of a complex hyperbolic manifold.*
- (2) *any quaternionic-Kähler manifold from $\mathcal{M}_{n,-4-\epsilon,-1,\pi}$ is diffeomorphic to a scalar multiple of a quaternionic hyperbolic manifold.*

Proof. Arguing by contradiction find a sequence M_k of finite volume Kähler or quaternionic-Kähler manifolds with fundamental group isomorphic to π and sectional curvatures within $[-4-1/k, -1]$. By 1.1, we can assume that (M_k, g_k) converges in $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ -Riemannian manifold (M, g) .

First, assume that each M_k is quaternionic-Kähler. Any quaternionic-Kähler is Einstein so M is smooth Einstein manifold and convergence is in C^∞ topology. Hence, sectional curvatures of M_k converge to sectional curvatures of M , so M is quarter-pinched, and we are done by [Yeu95].

Second, assume that each (M_k, g_k) is Kähler. Adapting the argument in [Ber60] to negative curvature, we deduce that the holomorphic sectional curvatures of M_k converge to -1 uniformly on M_k . Look at the "curvature 4-tensor" R_k^0 for Kähler metric of holomorphic sectional curvature -1 defined in terms of g_k and almost complex structure J_k of M_k [KN69, IX.7, right before 7.2]. Since sectional curvature can be written in terms of holomorphic sectional curvature [BG64] and the curvature 4-tensor can be written in terms of sectional curvature, the 4-tensor R_k of g_k is getting close to R_k^0 uniformly on M when $k \rightarrow \infty$. Taking traces we conclude that Ricci tensor of g_k is getting close to $-(n+1)g_k/2$ (see [KN69, IX.7.5]). Now g_k subconverges to a $C^{1,\alpha}$ -Riemannian metric on a finite volume manifold M hence $-(n+1)g_k/2 - Ric(g_k)$ converges to zero. Then the proof of [Gao90, theorem 0.4] implies that the limiting metric g is a weak solution of the Einstein equation, hence g is a C^∞ Einstein metric. Also (M, g) has Alexandrov curvature within $[-4, -1]$ hence $sec(M, g) \in [-4, -1]$ and we are done by [Yeu95]. \square

Remark 7.5. It is not clear whether the assumptions of 7.4 are necessary. However, there do exist compact negatively curved Kähler 4-manifolds which are not homotopy equivalent to locally symmetric manifolds [MS80]. Also in higher dimensions there are examples of compact almost quarter pinched Riemannian manifolds which are homeomorphic but not diffeomorphic to complex hyperbolic manifolds [FJ89a].

References

- [Ale57] A. D. Alexandrov, *Über eine Verallgemeinerung der Riemannschen Geometrie*, Schriften Forschungsinst. Math (1957), no. 1, 33–84, MR:19,304h.
- [And90] M. T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), no. 2, 429–445.
- [BCG95] G. Besson, G. Courtois, and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geom. Funct. Anal. **5** (1995), no. 5, 731–799. **5** (1995), no. 5, 731–799.
- [BCG98] G. Besson, G. Courtois, and S. Gallot, *Lemme de Schwarz réel at applications géométriques*, preprint, 1998.
- [BE98] Ž. Bižaca and J. Etnyre, *Smooth structures on collarable ends of 4-manifolds*, Topology **37** (1998), no. 3, 461–467.
- [Bel98a] I. Belegradek, *Counting open negatively curved manifolds up to tangential homotopy equivalence*, J. Differential Geom. **50** (1998), no. 1, 59–88.

- [Bel98b] I. Belegradek, *Pinching, Pontrjagin classes, and negatively curved vector bundles*, preprint, 1998.
- [Bel99] I. Belegradek, *Lipschitz precompactness for closed negatively curved manifolds*, Proc. Amer. Math. Soc. **127** (1999), no. 4, 1201–1208.
- [Ber60] M. Berger, *Pincement riemannien et pincement holomorphe*, Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 151–159.
- [Bes88] M. Bestvina, *Degenerations of the hyperbolic space*, Duke Math. J **56** (1988), 143–161.
- [BF95] M. Bestvina and M. Feighn, *Stable actions of groups on real trees*, Invent. Math. **121** (1995), no. 2, 287–321.
- [BG64] R. L. Bishop and S. I. Goldberg, *Some implications of the generalized Gauss-Bonnet theorem*, Trans. Amer. Math. Soc. **112** (1964), 508–535.
- [BGS85] W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Progress in mathematics, vol. 61, Birkhauser, 1985.
- [Bow93] B. H. Bowditch, *Discrete parabolic groups*, J. Diff. Geom. **38** (1993), 559–583.
- [Bow95] B. H. Bowditch, *Geometrical finiteness with variable negative curvature*, Duke Math. J. **77** (1995), 229–274.
- [Bow98] B. H. Bowditch, *Peripheral splittings of groups*, preprint, 1998.
- [Bro82] K. S. Brown, *Cohomology of groups*, Springer-Verlag, 1982.
- [BS87] M. Burger and V. Schroeder, *Amenable groups and stabilizers of measures on the boundary of a Hadamard manifold*, Math. Ann. **276** (1987), no. 3, 505–514.
- [CG85] J. Cheeger and M. Gromov, *On the characteristic numbers of complete manifolds of bounded curvature and finite volume*, Differential geometry and complex analysis, Springer, 1985, pp. 115–154.
- [Che55] S. S. Chern, *On curvature and characteristic classes of a Riemann manifold*, Abh. Math. Sem. Univ. Hamburg **20** (1955), 117–126, in book: Shiing-Shen Chern selected papers, Volume II, Springer-Verlag, 1989.
- [Dol72] A. Dold, *Lectures on algebraic topology*, Grundlehren der mathematische Wissenschaften 200, Springer-Verlag, 1972.
- [FJ89a] F. T. Farrell and L. E. Jones, *Complex hyperbolic manifolds and exotic smooth structures*, Invent. Math. **117** (1989), no. 1, 57–74.
- [FJ89b] F. T. Farrell and L. E. Jones, *Negatively curved manifolds with exotic smooth structures*, J. Amer. Math. Soc. **2** (1989), no. 4, 899–908.
- [FJ93a] F. T. Farrell and L. E. Jones, *Nonuniform hyperbolic lattices and exotic smooth structures*, J. Differential Geom. **38** (1993), no. 2, 235–261.

- [FJ93b] F. T. Farrell and L. E. Jones, *Topological rigidity for compact non-positively curved manifolds*, Proc. Sympos. Pure Math., 54, Part 3, Amer. Math. Soc., 1993, pp. 229–274.
- [FJ98] F. T. Farrell and L. E. Jones, *Rigidity for aspherical manifolds with $\pi_1 \subset \mathrm{GL}_m(R)$* , Asian J. Math. **2** (1998), no. 2, 215–262.
- [FJO98] F. T. Farrell, L. E. Jones, and P. Ontaneda, *Hyperbolic manifolds with negatively curved exotic triangulations in dimensions greater than five*, J. Differential Geom. **48** (1998), no. 2, 319–322.
- [FT95] M. H. Freedman and P. Teichner, *4-manifold topology. I. Subexponential groups*, Invent. Math. **122** (1995), no. 3, 509–529.
- [Fuk84] K. Fukaya, *A finiteness theorem for negatively curved manifolds*, J. Differential Geom. **20** (1984), no. 2, 497–521.
- [Fuk86] K. Fukaya, *Theory of convergence for Riemannian orbifolds*, Japan. J. Math. (N. S.) **12** (1986), no. 1, 121–160.
- [Gao90] L. Z. Gao, *Convergence of Riemannian manifolds; Ricci and $L_{n/2}$ -curvature pinching*, J. Differential Geom. **32** (1990), no. 2, 349–381.
- [Gro78] M. Gromov, *Manifolds of negative curvature*, J. Differential Geom. (1978), no. 2, 223–230.
- [Gro82] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–99.
- [GT87] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987), 1–12.
- [Gui92] C. R. Guilbault, *An open collar theorem for 4-manifolds*, Trans. Amer. Math. Soc. **331** (1992), no. 1, 227–245.
- [HH77] E. Heintze and H.-C. Im Hof, *Geometry of horospheres*, J. Differential Geom. **12** (1977), no. 4, 481–491.
- [HM74] M. Hirsch and B. Mazur, *Smoothings of piecewise linear manifolds*, Princeton University Press, Princeton, N. J., 1974, Annals of Mathematics Studies, No. 80.
- [KN69] S. Kobayashi and K. Nomidzu, *Foundations of differential geometry, vol. II*, Interscience Publishers, a division of John Wiley & Sons, 1969.
- [KS77] R. S. Kirby and L. S. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Annals of Mathematics Studies, No. 88, Princeton University Press, 1977.
- [LeB95] C. LeBrun, *Einstein metrics and Mostow rigidity*, Math. Res. Lett. **2** (1995), no. 1, 1–8.
- [MS80] G. D. Mostow and Y. T. Siu, *A compact Kähler surface of negative curvature not covered by the ball*, Ann. of Math. **112** (1980), 321–360.

- [Nik91] I. G. Nikolaev, *Closure of the set of classical Riemannian spaces*, J. of Soviet Math. **55** (1991), no. 6, 2100–2115.
- [Pau88] F. Paulin, *Topologie de Gromov équivariante, structures hyperboliques et arbres réels*, Invent. Math. **94** (1988), no. 1, 53–80.
- [Pau91] F. Paulin, *Outer automorphisms of hyperbolic groups and small actions on \mathbb{R} -trees*, Arboreal group theory (R. C. Alperin, ed.), Math. Sci. Res. Inst. Publ., 19, Springer, 1991, pp. 331–343.
- [Pra76] G. Prasad, *Discrete subgroups isomorphic to lattices in semisimple lie groups*, Amer. J. Math. **98** (1976), no. 1, 241–261.
- [Rei71] W. Reid, *Ordinary differential equations*, John Wiley & Sons Inc., 1971.
- [RS94] E. Rips and Z. Sela, *Structure and rigidity in hyperbolic groups I*, Geom. Funct. Anal. **4** (1994), no. 3, 337–371.
- [Sch84] V. Schroeder, *Finite volume and fundamental group on manifolds of negative curvature*, J. Differential Geom. **20** (1984), no. 1, 175–183.
- [Ser80] J.-P. Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980.
- [Wal68] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. Math. **87** (1968), 56–88.
- [Yeu95] S.-K. Yeung, *Uniformization of $1/4$ -pinched negatively curved manifolds with special holonomy*, Internat. Math. Res. Notices (1995), no. 8, 365–375.